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## PREFATORY NOTE

The purpose of this monograph is to present in a consecutive form the principal features of abstract and substitution group theory. The development of this branch of mathematics has been very rapid, especially in the last few years, and consequently there is much of great value to be found in a more or less fragmentary form throughout the various mathematical journals that has yet to be collected and discussed. It has been my purpose to examine in detail all memoirs dealing directly with such group theory (excluding, in particular, that of linear groups) and to construct from this material a continuous treatise on the subject.

In order to secure uniformity of terminology and method I may seem to have taken liberties with the work of others, I trust that such will not be found unwarranted. The amount of space at my disposal has compelled me to omit all proofs; instead I have given with each theorem or definition a reference to nearly every source, original or secondary. Corrections or additions will be most gratefully appreciated.

It is proper to add that as I was granted a Harrison Senior Fellowship to obtain opportunity for revising this work, its present form differs materially from that presented in May of 1901 to the Faculty of Philosophy of the University of Pennsylvania as a dissertation in partial fulfilment of the requirements for the degree of Doctor of Philosophy. The original draft was devoted to substitution theory alone and contained full proofs, with references to only the principal treatises.

* * *

To the authorities of the Columbia College Library and the Munich Staatsbibliothek I wish to express my acknowledgment of many courtesies extended to me. My deepest thanks are due to Dr. G. H. Hallett, under whose direction this dissertation was written and to whose inspiration it owes whatever merit it may possess. And I wish to acknowledge my especial gratitude to Prof. E. S. Crawley for his continual assistance and encouragement, without which this work would have been impossible.

BURTON SCOTT EASTON.

October 22, 1902.

# ABSTRACT GROUPS AND SUBSTITUTION GROUPS

## LIST OF PERIODICALS

AB Bulletin of the American Mathematical Society. New York.

AF Association française pour l'avancement des sciences. Paris Comptes Rendus. Paris.

AJ American Journal of Mathematics. Baltimore.

AT Transactions of the American Mathematical Society. New York.

AcM Acta Mathematica. Stockholm.

AiM Annali di matematica pura ed applicata. (Known as Brioschi's "Annali.") Milan.

AmM American Mathematical Monthly. Springfield, Missouri.

AnM Annals of Mathematics. Cambridge (Mass.).

Arc Archiv der Mathematik und Physik. ("Grunert's" or "Hoppe's.") Leipsic.

BB Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften. Berlin. ("Monatsberichte" prior to 1882.)

CJ (Crelle's) Journal für die reine und angewandete Mathematik. Berlin.

CP Mathematical papers read at the International Mathematical Congress held in connection with the World's Columbian Exposition, Chicago, 1893. New York.

CR Comptes rendus hebdomadaires des séances de l'Académie des Sciences. Paris.

**DB** (Darboux') Bulletin des sciences mathématiques. Paris.

**DV** Jahresbericht der Deutschen Mathematiker-Vereinigung. Leipsic.

**ET** Educational Times. London.

**EW** Encyclopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Leipsic.

**ÉcN** Annales scientifiques de l'École Normale Supérieure. Paris.

**ÉcP** Journal de l'École Polytechnique. Paris. (References to *cahiers.*)

**FB** Bulletin de la Société Mathématique de France. Paris.

**FM** Jahrbuch über die Fortschritte der Mathematik. Berlin.

**GM** Giornale di matematiche di Battaglini. Naples.

**GN** Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

**IM** Memorie della società italiana delle scienze. Modena.

**LJ** (Liouville's) Journal de Mathématiques pures et appliquées. Paris.

**LP** Proceedings of the London Mathematical Society.

**LnM** Memorie della Reale Accademia dei Lincei. Rome.

**LnR** Atti della Reale Accademia dei Lincei. Rendiconti. ("Transunti" prior to 1885.) Classe di scienze fisiche, matematiche e naturali. Rome.

**M** The Messenger of Mathematics. London.

**MA** Mathematische Annalen. Leipsic.

**Mh** Monatshefte für Mathematik und Physik. Vienna.

**NA** Nouvelles Annales de mathématiques. Paris.

**NY** Bulletin of the New York Mathematical Society.

**NpR** Rendiconti dell'Accademia delle scienze fisiche e mathematiche. Naples.

**PM** The London, Edinburgh, and Dublin Philosophical Magazine. London.

PlR Rendiconti del Circolo Matematico del Palermo.

PRS Proceedings of the Royal Society of London.

QJ The Quarterly Journal of pure and applied mathematics. London.

TlA Annales de la Faculté des Sciences de Toulouse pour les sciences mathématiques. Paris.

TlM Mémoires de l'Académie des sciences, inscriptions et belles lettres de Toulouse. Toulouse.

ToA Atti della Reale Accademia di Torino. Turin.

ToM Memorie della Reale Accademia delle scienze di Torino. Turin.

ZM Zeitschrift für Mathematik und Physik. ("Schlömilch's.") Leipsic.

## EXPLANATION OF REFERENCES

Papers are arranged in this bibliography in the order in which they were written (not published) by the author. The dates are those of publication. A † denotes that the paper has been examined in a review, a †† that only the title is known to the writer. Otherwise all papers have been examined. [ ] inclose reprints or references to reprints. An * denotes that a theorem referred to is incomplete or not quite identical with that given. All references are to pages,—e. g., Bg2, 220 is a reference to page 220 of Bagnera's second paper.

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## 1. SUBSTITUTIONS

1. The operation by which the members of a set of quantities are replaced by the same quantities in the same or a different order is termed a *substitution*.

Ch 1, 4; A 2, 76; Ch 3, 152; Ch 4, 594 [280]; Bt 2, 51; J 1, 119; Bc, 210; S, c, 219; S, e, 243; J 11, 141; J, 21; Ja 1, 280; Pt, 231; N, x, 227; N, 13; NC, 12; BD, 255; V, 1; Pc, 421; W, c, 514; E, 102; B, 1; Bi, 3; Bu, 209; N, a, 257.

2. The quantities on which the substitution is performed are termed the letters, elements, or marks of the substitution and their number is its *degree* (J11, 141), the number of marks replaced by marks other than themselves its *class* (N2, 50; NC, 85).

3. If the substitution $S$ replaces the marks $x_1, x_2, \cdots x_n$ by the marks $y_1, y_2, \cdots y_n$ (coinciding with the $x$'s in some order), then $S$ may be written in the form

$$\begin{pmatrix} x_1, & x_2, & \cdots & x_n \\ y_1, & y_2, & \cdots & y_n \end{pmatrix}.$$

Ch 1, 4; A 2, 75; Bt 2, 51*; Bc, 230; N, 19; NC, 19; V, 3; W, c, 514; E, 103; B, 1; Bu, 209; N, a, 257. Inverted in Ch 3, 152; Ch 4, [281]; Bt 3, 6; S, c, 220; S, e, 244; Pt, 231; N, x, 227; Pc, 421; Bi, 4.

4. The number of substitutions possible on $n$ marks is $n!$, if the (*identical*) substitution replacing each mark by itself is included.

5. The number of substitutions of degree and class $n$ is the integer nearest $n! \div e$, where $e$ is the Napierian base. (F2, 292).

6. The effects of a substitution $S_1$ followed by a second $S_2$ can be accomplished always by some single substitution $S_3$, termed the *product* of $S_1$ and $S_2$, and written $S_3 = S_1S_2$.

Ch 1, 10; Ch, 3, 153; Ch, 4, [282]; J 1, 120; Bc, 211; S, c, 221; S, e, 245; J 11, 141; J, 21; Pt, 231; N, x, 230; N, 24; NC, 23; BD, 255; V, 3; Pc, 422; W, c, 515; E, 108; B, 4; Bi, 5; Bu, 210; N, a, 257.

Note 1. The commutative law does not hold in general for such products.—if $ST = TS$, then $S$ and $T$ are termed *permutable* substitutions. But for three or more factors the associative law holds.

Note 2. S, Pc, and many writers on the analytic representation of substitutions reverse the order of the factors.

7. A substitution that effects merely a cyclic interchange of the marks is termed *circular*.

Ch 1, 17; Ch 3, 157; Ch 4, [285]; S 3, 50; Bt 2, 53; Ht 1, 331; J1, 121; S, c, 226; S, e, 250; J, 21; Pt. 235; N, 34; NC, 35; BD, 262; V, 2; Pc, 422; W, c, 534; E, 153; B, 7; Bi. 8; Bu, 210; N, a, 262.

Such a substitution on, *e. g.*, the arrangement $x_1, x_2, \cdot\cdot x_n$ is denoted by $(x_1, x_2, \cdot\cdot x_n)$.

8. Any substitution may be written as the product of circular substitutions, no two of which have a mark in common and which are termed *cycles* of the given substitution.

Ch 2, 38; Ch 3, 159; Ch 4, [286]; Brt, 125, Bc 217; S, c, 226; S, e, 250; J, 21; Pt, 236; N, x, 227; N, 20; NC. 19; BD, 263; V, 2; Pc, 422; W, c, 534; E, 156; B, 2; Bi, 9; Bu, 210; N, a, 262.

9. Let $s$ be the smallest positive value (not zero) of $x$ for which $S^x = 1$. Then $s$ is termed the *order* of $S$.

Ch 1, 13 ("degree"); Ch 3, 157 (do); Ch 4, [284] (do); Bt 2, 54 (do); A 2, 76; J1, 121*; Bc 211; S, c, 224; S, e, 248; J, 22; Pt, 234; N, x, 231; N, 36; NC, 36; BD, 264; V, 4; W, c, 542 ("degree"); E, 163; B, 6; Bi, 7; Bu, 210; N, a, 258.

Note 1. $S^{-x}$ may be defined as $S^{s-x}$. In particular $S^{-1}$ is the *inverse* of $S$.

Note 2. $(ST)^{-1} = T^{-1}S^{-1}$.

Note 3. The order of a substitution equals the l. c. m. of the orders of its cycles.

Note 4. In this memoir the order of $S_1$ will be denoted by $s_1$, that of $T_a$ by $t_a$, etc.

10. $S^x$ is of order $x : dv(x, s)$. It contains $dv(x, s)$ cycles if $S$ is circular.

S, c, 224; S, e, 248; J, 22*; Pt, 237; N, 36; NC, 36; B, 15.

11. The order of the product of two permutable substitutions is not greater than the l. c. m. of their orders, but may be less. (Full discussion in M 77.) The product of two not-permutable substitutions may be of any order. (M 62, M 72.)

12. If the cycles of a substitution have all the same order the substitution is termed *regular*.

Ch 3, 202; Ch 5, [342]; J 1, 123; S, c, 229; S, e, 253; Pt, 237; E, 164; B, 7; Bu, 211.

13. Every regular substitution is a power of a circular substitution.

S, c, 230; S, e, 254; Pt, 238; E, 164; B, 8.

14. If two substitutions differ only in the designation of their marks they are termed *similar*.

Ch 3, 165; Ch 4, [288]; Ch 5, [348]; S, c, 233; S, e, 257; J, 23; Pt, 239; N, 47; NC, 49; BD, 263; V, 10; Pc, 460; E, 168; B, 7; Bi, 13; Bu, 211; N, a, 307.

15. The substitution $T^{-1}ST$ is termed the *transformed* of $S$ by $T$.

Bt 2, 55 ("derived"); Bt 3, 6 (do); J 5, 110; J 11, 141; J, 23; Pt, 239; N, x, 242; N, 47; NC, 49; BD, 273; V, 11; Pc, 426*; W, c, 548; E, 176; B, 27; Bi, 13; Bu, 211; N, a, 307.

16. $T^{-1}ST$ is obtained by "operating $T$ into the cycles of $S$,"—*i. e.*, by operating $T$ on the marks of $S$ as they stand in their cycles. Whence $S$ and $T^{-1}ST$ are similar.

J 1, 125; S, c, 233; S, e, 257; J, 23; Pt, 239; N, x, 242; N, 47; NC, 50; BD, 274; V, 11; Pc, 460; W, c, 536; E, 176; B, 7; Bi, 13; Bu, 211.

Note 1. A substitution can be transformed into any substitution similar to it. (Ch 3, 169.)

Note 2. $ST$ and $TS$ are similar. (Ch 4, [324]*.)

17. A substitution of class 2 is termed a *transposition.*

Ch 1, 18; A 2, 79; S, c, 227; S, e, 251; J, 61; Pt, 236; N, x, 227; N, 14; NC, 13; V, 2; Pc, 421; W, c, 79; E, 140; B, 9; Bi, 10; Bu, 213; N, a, 258.

18. Any substitution can be written as a product of transpositions and in an infinite number of ways, the number of transpositions, however, being always even or always odd. Hence substitutions may be classified as *even* or *odd.*

Ch 2, 41; Ki 3, 225; S, c, 250; S, e, 277; J, 61; Pt, 244; N, x, 237; N, 18; NC, 17; BD, 269; V, 7; Pc, 430*; W, c, 537; E, 146; B, 10; Bi, 12; Bu, 213; N, a, 260.

"First" and "second" "classes" in Ch, S, W, "positive" and "negative" in lists of groups (due to C.).

19. An even substitution is a product of circular substitutions of class 3.

Wn, 65; J, 63; H 4, 62; V, 7; W, c, 538; E, 248; B, 10.

20. If $S$ contains $m_i$ cycles of order $n_i$ there are $M = m_1!\, m_2! \cdots n_1^{m_1} \cdot n_2^{m_2} \cdots$ substitutions formed from the marks of $S$ that are permutable with it.

Ch 9; J 1, 129; Cp 1, 66 and 70–74; S, e, 270; E, 494; B, 216.

21. If $n$ is the class of $S$ there are $N = n! : M$ substitutions similar to $S$.

Ch 4, [289]; Ch 8; S, c, 236; S, e, 260; N, 22; NC, 22; BD, 264.

22. On the general theory of permutable substitutions see Brn; Dz.

## 2. GROUPS

23. Let $S_1$, $S_2$, $\cdots$ $S_g$ represent a finite number of distinct things, subject to the following conditions:

1. By some prescribed law the combination in a prescribed order of two things (identical or distinct) of the set is to be regarded as equal to some thing of the set.

2. If more than two things are combined the associative law is obeyed.

3. If $S_cS_a = S_cS_b$ (or if $S_aS_c = S_bS_c$) then $S_a = S_b$.

The things fulfilling these conditions are said to form a *finite discontinuous group.*

The things forming the group are commonly termed *elements* and may be regarded as *operations,* their combination will be regarded as their *product.*

The following consequences of the definitions result immediately:

1. The equation $S_aS_b = S_c$ can be satisfied whenever any two of these three elements are given and the third is to be found.

2. Every group contains an element $S_e$ such that $S_aS_e = S_eS_a = S_a$, the *identical* element, written as 1.

3. To every element $S_a$ of a group there corresponds an element $S_a^{-1}$ such that $S_aS_a^{-1} = S_a^{-1}S_a = 1$, $S_a^{-1}$ being termed the element *inverse* to $S_a$.

The above definition of a group is the development of about a century. While the implicit concept of a group is as old as mathematical thought (cf. Pn), the idea appears explicitly first in Lg and Vd, while the first definite enunciation of it seems to be that in Rf, a, 251. But the proper founder of group theory is Évariste Galois, who formulated the following definition:

A set consisting of a finite number of substitutions, such that the product of any two (identical or distinct) of the set equals a substitution of the set, is termed a *group* of substitutions.

G 3, 35; Ch 3, 183; Ch 4, [290]; Bt 2, 59; C 4, [596]; J 1, 120; Bc, 212; S, c, 251; S, e, 278; J 3, 770; J 5, 109; J 11, 141; J, 22; Ja 1, 284; Pt, 245; N, x, 231; N, 26; NC, 25; Kl, 5; BD, 257; V, 4; Pc, 424; W, c, 513, 516; E, 189; B, 12; Bi, 14; Bu, 211; N, a, 258.

The separation of abstract group theory from that of substitution groups is not easy to mark. J 14, however, is an elaborate discussion of abstract groups and the earlier memoirs of Klein dealt with groups other than substitution groups although closely connected with them. The honor of the first explicit definition of abstract groups is, however, due to Cayley, with this dictum, "A group is defined by means of the laws of combinations of its symbols" (C 13, 51; C 15, 127), published in 1878. (Cf. C 1, [124].) The definition of a group given in connection with this dictum differs only slightly from the one in this section, formulated by F 1, 180. Cf. H 1, 29; H 5, 305; BD, 296; W, d, 3; Bi, 33; Bu, 218; Zs, 183; N, a, 235. A slightly less general definitions is given by Dy 3, 73; B, 11. For very exact definitions see Hn 1; Mo 11; Hn 2.

Certain terms applied to substitutions—as order, transformation—will be used for the elements of a group without repeating the definitions.

24. The number of elements in a group is the *order* of that group.

Rf, a, 251 ("degree of equality"); Ch 1, 7 ("indicative divisor"); Ch 3, 183 ("order"); Ch 4, 605; Bt 2, 59 ("degree"); J 1, 120; S, c, 251; S, e, 278; J 11, 141; J, 22; Pt, 245; N, x, 231; N, 26; NC, 26; Kl, 6 ("degree"); BD, 260; V, 4; Pc, 424; W, c, 517 ("degree"); W, d, 4 (do.); E, 198; B, 14; Bi, 15; Bu, 212; N, a, 236.

Note. It is customary to denote the order of a group by a subscript applied to its symbol,—as $G_{24}$. And it is very convenient to consider $G$ of order $g$, $H'$ of order $h'$, etc.

25. Let two groups have the same order. If between their elements there can be established a one-one correspondence such that the product of two elements of one corresponds to the product of the corresponding two elements of the other, the groups are said to be abstractly identical or *simply isomorphic*.

J 9, 836; J, 56; Cp 3, 33; Kl, 8; N, 97; NC, 83; BD, 259; Pc, 452; W, c, 548; W, d, 6; B, 22; Bi, 31; Bu, 217; N, a, 339.

D writes $G \sim G'$.

26. If all the elements of a group are permutable the group is termed *abelian*.

K 6, 882; BD, 277; W, c, 517; W, d, 6; B, 46; Bi, 69, Bu, 222; N, a, 239.

27. A group composed of the powers of an element is termed *cyclic*.

Rf, a, 247 ("simple permutation"); N, 134; NC, 145; V, 45; W, c, 517; W, d, 11; B, 25; Bi, 15; Bu, 214; N, a, 264; and implicitly in every other writer on groups.

28. If some of the elements of $G$ form a group in themselves, $H$ is a *divisor* or *subgroup* of $G$.

*Divisor* in G 4, 58; C 2, [132]; FS, 220; W, c, 517; W, d, 7; N, a, 236.

*Subgroup* in N, 44; NC, 40; BD, 269; V, 9; Pc, 448; B, 25; Bi, 15.

Unnamed in Bc, 212; S, c, 251; S, e, 278; J, 25; Pt, 245; N, x, 240.

$H$ will be always a subgroup of $G$ in this memoir (Frobenius reverses this conventional use).

29. If $H$ is a subgroup of $G$, $h$ is a divisor of $g$.

G 5, 26; G 4, [299]; J 1, 166; Bc, 212; S, c, 251; S, e, 278; J 11, 142; J, 25; Pt, 245; N, x, 243; N, 45; NC, 47; BD, 271; V, 9; Pc, 448; W, c, 544; W, d, 10; E, 203; B, 25; Bi, 16; Bu, 212; N, a, 238.

30. The notation $G = H + S_2H + S_3H + \cdot + S_nH$ was introduced by G. And the set of elements $S_aH$—which is not a group—is termed an *adjunct group* of $H$. (W, d, 8; N, a, 306 ("conjugate complex).")

31. The order of every element of $G$ divides $g$.

Ch 3, 185; Ch 4, [291]; Bt, 2, 60. Others as above.

32. The ratio $g : h$ is termed the *index* of $H$ under $G$, and is denoted by $(G, H)$. Dd, 475; W, d, 8; Bu, 212. Others as in no. 49.

33. The elements common to two groups $G$ and $G'$ form a group $H$ that is a subgroup of both $G$ and $G'$.

S, c, 254 ; S, e, 281 ; Pt, 247 ; N, 45 ; NC, 47 ; FS, 220 ; BD, 276 ; Pc, 449 ; W, c, 551 ; W, d, 10 ; E, 213 ; B, 26 ; B, 18.

*H* is known as the *cross-cut* or *greatest common divisor* of *G* of *G'* and may be written as *dv*(*G*, *G'*).

34. All elements permutable with a given element form a group.

Ch 3, 225 ; S, c, 260 ; S, e, 287 ; C 9, [47] ; Pt, 247 ; N, 85 ; NC, 80 ; BD, 277 ; V, 12 ; E, 209 ; B, 29 ; Bi, 18 ; N, a, 308.

35. If $G$ contains $S$ and $h-1$ other elements permutable with $S$, then $G$ contains $g/h$ elements conjugate to $S$. (Two elements of a group are conjugate in the group when one can be transformed into the other by an element of the group.) (B, 29 ; N, a, 309.)

36. All elements transforming a given element into its powers form a group.

J 1, 130; S, c, 261; S, e, 288; Pt, 248; N, 48; NC, 51; E, 215.

37. By combining in every possible way a number of given elements and groups $S_1$, $S_2$, $\cdots$, $G_1'$. $G_1''$ $\cdots$, a group is formed that is said to be *generated* by $S_1$, $S_2$, $\cdots$, $G'$, $G''$, $\cdots$, and which is written as $\{S_1, S_2, \cdots, G', G'', \cdots\}$.

Ch 3, 183 ; Ch 4, 605 ; Bc, 220 ; all later writers. In particular Dy 2.

38. From every group it is possible to choose a set of elements such that the group is generated by them, but none of the set can be expressed in terms of the others. These are termed *generating elements*. (J 1, 120 *; Bc, 220.) These generating elements define the group completely and the relations between them are termed the *equations* of the group.

Certain cases are studied in Bru 1 ; Bru 2 ; Fl ; M 82 ; M 83 ; M 85 ; M 86 ; M 88.

By the theory of these generating elements several methods have been devised for the graphical representation of groups. C 13 ; C 15 ; C 16 ; C 18 ; Dy 2 ; Mo 7, 273 seq ; Kp, 18 ; B 1 ; B 2 ; Ms ; B, 255 seq., 306 seq.; Hf 2 ; Fk 1.

39. Let every element of $G'$ be permutable with every element of $G''$, — *i. e.*, let $G'$ and $G''$ be *absolutely permutable.* Then, if $dv(G', G'') = 1$, the group $\{G', G''\}$, of order $g'g''$, is termed the *direct product* of $G'$ and $G''$.

C 2, [132] ; C 21, 62 ; H 1, 36 ; H 5, 330 ; BD, 299 ; B, 40 ; Bu, 219 ; N, a, 249.

40. A group that is the direct product of two of its subgroups is termed *divisible* ("zerfallend").

Dy 1, 482 ; FS, 221 ; H 1, 36 ; H 5, 335 ; Bu, 219 ; N, a, 249.

41. Let $G = 1, S_2, \cdots S_g$ and $G' = 1, S_2', \cdots S_{g'}'$. If every product of the form $S_a S_b'$ can be written equal to one of the form $S_c' S_d$ and conversely, the groups are termed *permutable,*—written $GG' = G'G$.

Ch 3, 227 ; Ch 5, [361] ; S, c, 256 ; S, e, 283 ; Pt, 253 ; V 12* ; Pc, 455* ; E, 218 ; B, 41 ; N, a, 327.

42. If $G'G'' = G''G'$ and $dv(G', G'') = 1$, then $\{G', G''\}$ is of order $g'g''$.

Ch 3, 229 ; Ch 4, [327]* ; Ch 5, [364] ; S, c, 265 ; S, e, 292 ; Pt, 253 ; N, 39 ; NC, 38 ; E, 219.

43. If $G'G'' = G''G'$ and $dv(G', G'') = H$, then $\{G', G''\}$ is of order $g'g'' : h$.

Bc, 226 ; Gi 1, 222 ; B, 42 ; N, a, 328. Less generally in Cp 3, 79 ; Br, 232 ; V, 14 ; Pc, 455 ; Bi, 39.

## 3. SUBSTITUTION GROUPS

44. The total number of distinct marks occurring in all the substitutions of a substitution group is termed the *degree* of that group.

J 11, 141 ; J, 22 ; Ja, 284 ; N, x, 231 ; N, 26 ; NC, 25 ; BD, 260 ; V, 4 ; Pc, 424 ; B, 138 ; Bu, 212.

Note 1. The degree of $G$ is denoted by a superscript,—as $G^{48}$.

Note 2. In this memoir $G$ will be regarded as of degree $n$.

**45.** If $G$ contains a substitution of class $k$ but none (excepting 1) of lower class, $G$ is said to be of class $k$.

J 21, 408 [854]; J 22, 853; N 5, 337; Bu, 214.

**46.** The group formed by all the ($n$!) substitutions possible on $n$ marks is termed the *symmetric* group of degree $n$.

Ch 3, 183; S, c, 251; S, e, 278; J, 26; N, x, 236; N, 33; NC, 32; BD, 257; V, 5; Pc, 424; W, c, 533; E, 192; B, 139; Bi, 15; Bu, 213; N, a, 258.

**47.** This group is defined abstractly by the equations: — $S_1^2 = S_2^2 = \cdots = S_{n-1}^2 = 1$, $S_iS_j = S_jS_i (i = 1, 2, \cdots n - 3, j = i + 2, i + 3, \cdots n - 1)$, $S_jS_{j+1}S_j = S_{j+1}S_jS_{j+1}$.

D, 287. Cf. Cp 5; B 13; Mo 6, 363, 364; D 19, 351; dS 1, 1031.

**48.** The order of every substitution group of degree $n$ divides $n$!.

Lg, 373*; Rf, a, 102*; Ab, 38 (first complete proof); A 2, 76; Ch 3, 184; Ch 4, [291]; S 2, 7; S, a; J1, 121; Bc, 212; S, c, 254; S, e, 280; J, 26; Pt, 246; N, x, 241; N, 44; NC, 46; BD, 270; V, 10; Pc, 427; E, 198; B, 139; Bi, 18; Bu, 212.

**49.** The ratio $n! : g$ is termed the *index* of $G$.

Ch 1, 6; S, c, 287; S, e, 314; Pt, 269; E, 201; B, 382; Bu, 212.

**50.** The group formed by all the ($\frac{1}{2} \cdot n$!) even substitutions possible on $n$ marks is termed the *alternating* group of degree $n$.

Ch 1, 19 (implicitly); Bc, 226; S, c, 257; S, e, 284; J 3, 773; J, 63; Pt, 251; N, x, 237; N, 33; NC, 33; BD, 269; V, 5; Pc, 430; W, c, 538; E, 192; B, 139; Bi, 15; Bu, 213; N, a, 261.

**51.** This group is defined abstractly by the equations: — $1 = S_1^3 = S_{i+1}^2 = (S_iS_{i+1})^3 = (S_iS_j)^2 (i, j = 1, 2, \cdots n - 2, j > i + 1)$.

D, 289. Cf. B 13, 129*; Mo 6, 366; D 26, 564; dS 1, 1031.

52. There is no other substitution group of degree $n$ and order $\frac{1}{2}\cdot n!$.

Ab; Mt 3, 19*; Bc, 227; S, c, 257; S, e, 284; Pt, 251; N, 34; NC, 34; E, 198; Bi, 19.

53. All or exactly half of the substitutions of every substitution group are even.

J, 62; N, 35; NC, 35; B, 140.

54. If a group of degree $n$ contain the substitutions (123), (124), ·· $(12n)$, it contains the alternating group of degree $n$.

S, c, 259; S, e, 286; N, 35; NC, 35; V, 7; W, c, 539 (with an extension); B, 10.

55. If a group of degree $n$ contain the substitutions (12), (13), ··· $(1n)$, it is symmetric.

Bc, 228; N, 33; NC, 33; V, 5; W, c, 534; B, 10.

56. If a group of degree $n$ contains all possible circular substitutions of degree $m$, it is the symmetric group or contains the alternating group according as $m$ is even or odd.

S, c, 259; S, e, 286; Pt, 252; N, 35; NC, 35; E, 251; N, a, 263.

57. If two substitution groups differ only in the designation of their marks they are termed *similar*.

S, c, 255; S, e, 282; N, 47; NC, 49; E, 211; N, a, 307. It is not sufficient that all the substitutions of one should be similar to all those of the other. Bz 1, 204; B 14; M 54.

58. For a compact (although somewhat complicated) notation for substitution groups see C 19; Co 7.

## 4. CONJUGACY

59. If all the elements of a group $G$ are transformed by an element $S$ (whether in $G$ or not) the resulting elements form a group. This group is termed the *transformed* of $G$ by $S$ and is written $S^{-1}GS$.

Bt 2, 61; S, c, 255; S, e, 282; J 11, 142; J, 24; Pt, 247;

N, 47; NC, 49; Kl, 7; BD, 273; V, 11; Pc, 426; W, c, 548; W, d, 11; B, 28; Bi, 35; N, a, 307.

60. If $S^{-1}GS = G$, $S$ is said to be *permutable* with $G$.

J 1, 162; J, 24; Pt, 247; N, 85; NC, 79; V, 12; E, 211; B, 29; Bi, 35; N, a, 326.

61. If $H$ and $H'$ are subgroups of $G$ and if $G$ contains an element $S$ such that $S^{-1}HS = H'$, then $H$ and $H'$ are termed *conjugate subgroups* ("gleichberechtigt").

Ki 2, 78 ("equivalent"); Dy 3, 76; Kl, 7; NC, 49; BD, 274; V, 12; W, c, 547; W, d, 11; B, 29; Bu, 218.

62. If every element of $G$ is permutable with $H$, $H$ is termed a *self-conjugate* or *invariant* subgroup. ("Ausgezeichnete Untergruppe," "Normaltheiler.")

G 5, 25; Bt 2, 65 ("proper divisor"); S, c, 621; S, e, 651; J, 41; Kg 2, 431; Dy 3, 76; N, 85; NC, 79; Kl, 7; F 2, 285 ("monotypic"); BD, 274; V, 12; Pc, 451; W, c, 553; W, d, 12; B, 29; Bi, 35; Bu, 219; N, a, 328 ("autojugate").

63. If a group contains no self-conjugate subgroup other than identity it is termed *simple*. Otherwise it is *composite*.

G5, 26 ("indécomposable"); Bt 2, 62 ("prime"); Ki 1, 283; Dp 1, 433 ("inséparable"); J 3, 772; J 11, 142; J, 41; Cp 3, 76; Pt, 248; N, 86; NC, 96; Kl, 7; BD, 294; V, 16; Pc, 451; W, c, 553; W, d, 13; B, 29; Bi, 36; Bu, 219; N, a, 328.

64. If $H'$ and $H''$ are self-conjugate subgroups of $G$, they are permutable, $dv(H', H'')$ is self-conjugate in $H'$, $H''$, and $G$, and $\{H', H''\}$ is self-conjugate in $G$ unless it coincides with $G$.

If $H'$ is self-conjugate in $G$ while $H''$ is a not-self-conjugate subgroup, then $dv(H', H'')$ is self-conjugate in $H'$ and $H''$ but not always in $G$.

If $H$ is self-conjugate in $G$ and $I$ self-conjugate in $H$, $I$ is not always self-conjugate in $G$.

The self-conjugate elements of a non-abelian group form a self-conjugate abelian subgroup — the *cogredient* subgroup. (M 49, 222.)

If $H$ is self-conjugate in $G$, $S^{-1}HS$ is self-conjugate in $S^{-1}GS$.

References for the above theorems seem unnecessary.

65. All elements permutable with a group $G$ form a group $F$, identical with $G$ or containing $G$ self-conjugately.

N, 86; NC, 80; V, 12*; BD, 277; B, 31.

66. If $F$ is a subgroup of a still larger group $E$, then $E$ contains $\frac{e}{f}$ subgroups conjugate to $G$. (B, 31.)

67. If $H'$ and $H''$ are each self-conjugate in $G$ and if $dv(H', H'') = 1$, then $H'$ and $H''$ are absolutely permutable. (No. 39.)

Dy 3, 97; H1, 35; F 4, 169; F 5, 982; B, 44.

68. Two simple groups of the same order are not necessarily isomorphic. This is proved by the existence of two non-isomorphic simple groups of order 20160. (St 1; St 4.) More generally in D 24; D 25.

69. For tests as to the simplicity of a given group see Dy 3, 77; Ft 2, 507; B, 34. With Kl, 18; Mo 3, 64 compare M5.

70. If an element (or subgroup) of a group be transformed by all the elements of $G$ in turn, the resulting distinct elements (or subgroups) are said to form a *complete* (*conjugate*) *set* in $G$. And all the elements of $G$ can be distributed into mutually exclusive complete sets.

F 1, 181; Dy 3, 75; W, d, 132; B, 31.

Note. On the basis of these complete sets has been constructed a theory of the utmost importance — that of group characteristics. See F8, and following papers, B 19, and following papers, W, d, 193 seq.

71. If $S_1, S_2, \cdots S_m$ is a complete set of elements or subgroups in $G$, then $\{S_1, S_2, \cdots S_m\}$ coincides with $G$ or is self-conjugate in $G$.

J 27, 176; Dy 3, 76; B, 34. And obviously.

72. If $H_1, \cdot\cdot H_m$ is a complete set of subgroups in $G$, then $dv(H_1, H_2, \cdots H_m)$ is self-conjugate in $G$.

V, 190; W, c, 553; B, 33. And obviously.

## 5. MULTIPLE ISOMORPHISM AND QUOTIENT GROUPS

73. Between $G$ and $G'$ let the following relations exist: 1. $g = mg'$ where $m$ is an integer. 2. To each element of $G'$ there correspond $m$ elements of $G$, such that if $S_a'S_b' = S_c'$ in $G'$ then the product of one of the $m$ elements corresponding to $S_a'$ in $G$ by one of those corresponding to $S_b'$ is an element corresponding to $S_c'$. 3. To each element of $G$ corresponds one and only one element of $G'$. Then $G$ and $G'$ are $(m-1)$-*fold* isomorphic.

J 9, 836; J, 56; Cp 3, 33; N, 97; NC, 83; Kl, 8; H1, 32; BD, 275; Pc, 452; W, d, 18; B, 36; Bi, 31; Bu, 217; N, a, 341.

I suggest the notation $G \sim (m) G'$.

74. Again, let: 1. $ng = mg'$. 2. As above. 3. To each element of $G$ there correspond $n$ of $G'$. Then $G$ and $G'$ are $(m-n)$-fold isomorphic. (Then $G(n) \sim (m) G'$.)

Cp 3, 33; BD, 290*; W, d, 19; B, 40.

75. Let $G$ contain a self-conjugate subgroup $H$ and let $g/h = \gamma$. Then a group $\Gamma$ can be constructed, of order $\gamma$ and such that $G \sim (h) \Gamma$. And 1 in $\Gamma$ will correspond to $H$ in $G$ and every other element of $\Gamma$ will correspond to an adjunct-group to $H$ in $G$.

G 3, 45; G 5, 25; Bt 2, 61*; J 9, 836*; J, 40; J 26, 46; Cp 3, 45; H 1, 31; NC, 96; Bz, 93; BD, 296; Pc, 452; W, d, 16; B, 38; Bi, 36; Bu, 219; N, a, 343.

76. $\Gamma$ is represented by the symbol $\frac{G}{H}$ (J 26, 46—cf. C 20) and is termed the *quotient-group* of $G$ and $H$.

F 4, 169 uses this symbol in a slightly extended sense — Bi, 36 applies it also to the case where $H$ is not self-conjugate. (Then $\frac{G}{H} \sim \frac{G}{K}$, — $K$ being the largest subgroup of $H$ that is self-conjugate in $G$.)

77. If $G \sim (h)\, G'$ and if the (self-conjugate) subgroup $H$ corresponds in $G$ to 1 in $G'$ then $\frac{G}{H} \sim G'$.

F 4, 168; W, d, 19; B, 38; others implicitly.

More generally, if $G(h') \sim (h)\, G'$ let $H'$ in $G'$ correspond to 1 in $G$. Then $\frac{G}{H} \sim \frac{G'}{H'}$. (F 4, 169.)

78. If $GG' = G'G$, let $F = \{G, G'\}$ while $K = dv(G, G')$. Then $\frac{E}{G} \sim \frac{G'}{K}$. (F 4, 169,—cf. no. 79, below.)

79. Let $H$ be a self-conjugate subgroup of $G$ and let $K$ be a self-conjugate subgroup of $G$ contained in $H$. Then $\frac{H}{K}$ is self-conjugate in $\frac{G}{K}$. (Reference needless.)

If $K$ is self-conjugate in $G$, let $\frac{G}{K}$ have a subgroup $M$. Then $G$ has a subgroup $H$ and $h = mk$. If $M$ is self-conjugate in $\frac{G}{K}$, $H$ is self-conjugate in $G$. (Do.)

80. Let $H$ be self-conjugate in $G$. If $H$ is a subgroup of no larger self-conjugate subgroup of $G$, it is termed a *maximal* self-conjugate subgroup of $G$.

J 11, 151; J, 41; N, 86; NC, 96; Kl, 7; H 1, 30; BD, 294; V, 16; Pc, 453; W, d, 23; B, 35; Bi, 36; Bu, 219; N, a, 331.

Then $\frac{G}{H}$ is simple. (Cp 3, 76.)

81. Let $G$ have the two maximal subgroups $H$ and $H'$

and let $dv(H, H') = K$. Then $K$ is a maximal self-conjugate subgroup of both $H$ and $H'$. And $\frac{g}{h} = \frac{h'}{k}$.

J 12, 141*; J, 43; Cp 3, 79; H 1, 36; N, 89; NC, 99; BD, 299; V, 17; Pc, 459; W, d, 28; B, 121; Bi, 42; N, a, 333.

82. Moreover $\frac{G}{H} \sim \frac{H'}{K}$ and $\frac{G}{H'} \sim \frac{H}{K}$.

H 1, 36; BD, 299; W, d, 28; B, 121; Bi, 48.

83. More generally, let $H$ be self-conjugate in $G$ and let $K$ and $K'$ be subgroups of $H$ that are each self-conjugate in $G$. And, while $K$ and $K'$ are not necessarily maximal in $H$, let there be no self-conjugate subgroup *of* $G$ that is contained in $H$ and that contains $K$ or $K'$. Then $\frac{H}{K} \sim \frac{K'}{L}$ and $\frac{H}{K'} \sim \frac{K}{L}$, where $dv(K, K') = L$. (N 1, 84*; B, 119.)

84. Let $H$ be maximal in $G$. Then $G$ can be generated by $H$ and a set of elements, conjugate in $G$, and which occur in $H$ only when raised to a prime power. (Consequently a simple group can be generated entirely by a conjugate set of elements of prime power.)

J 1, 192; N 1, 91; N, 91; NC, 101 (translation obscure).

## 6. COMPOSITION SERIES

85. Let $G, H, I, \cdots L$ be a series of groups, so chosen that each is a maximal self-conjugate subgroup of the preceding, while $L$ is simple. These are said to form a *composition series* of $G$, and the ratios $\frac{g}{h}, \frac{h}{i} \cdots l$ are termed the *composition factors.*

J 1, 192* ("échelle"); J 3 (implicitly); J 5, 112* ("échelle"); J 11, 152; J 12, 139; J 13, 257; J, 41; Cp 3, 83; N, 87; NC, 96; Ft 2, 498; Bz, 97; BD, 294; V, 16; Pc, 454; W, d, 23; B 118; Bi, 43; Bu, 220; N, a, 331.

86. The (simple) groups $\frac{G}{H}, \frac{H}{I}, \cdots \frac{L}{1}$ are termed the *factor groups* of $G$.

J 26, 48 ("composing groups"); H 1, 33; NC, 96; B 118; used by others unnamed.

87. The composition factors of a given group are the same (except as regards order) however the series is chosen.

Stated in J 11; J 12; J 13, as above. Proofs as above, also in N 1, 84; Br; P 4.

88. The factor groups of a given group are the same (except as regards order) however the series is chosen.

Cp 3, 79*; H 1, 38; BD, 297; F 4, 169; Br, 234; B, 122; Bi, 47; Bu, 220. Cf. Dv.

89. If $H$ is a subgroup of $G$ each composition factor of $H$ divides one of the composition factors of $G$. (J 12, 140.)

If $G \sim (h) G'$ and if $H$ in $G$ corresponds to 1 in $G'$, then the composition factors of $G$ are those of $G'$ combined with those of $H$. (Bi, 47.)

90. If $H$ is any self-conjugate subgroup of $G$, $H$ occurs in some composition series of $G$. (W, d, 30; Bi, 49.)

91. The alternating substitution group of degree $> 4$ is simple. Hence if $n > 4$, the symmetric group of degree $n$ has the composition factors 2 and $\frac{1}{2} \cdot n\,!$.

J 3, 773; J, 66; N 3, 87; K 8, 208; N, 91; NC, 102; Kg 1; Kl, 18; Bz, 98; BD, 315; V, 18; Pc, 462; W, c, 649; Bk 2, 581; B, 154; Bi, 54; Bu, 219; N, a, 311.

92. Let $G, H, I, \cdots L, 1$ be a series of groups so chosen that each is self-conjugate *in* $G$, while no self-conjugate group of $G$ exists that is contained in one of these groups and contains the next following. Then these groups are said to form a *chief-series* of $G$.

J, 48; J 19, 7; H 1, 38; W, d, 31; B, 123; Bi, 50; Bu, 220. Slightly different use in N 1, 84; N, 92; NC, 104; N, a, 336.

The invariancy of the chief composition-factors and factor groups is true in this case also (J, 663*—others), but not always with Netto's definition (cf. N, a, 337).

93. When a chief series is converted into an ordinary series by the insertion of other groups, let $H'H''$, $\cdots$ $H^a$ be inserted between $H$ and $K$ (e. g.). Then $\frac{H}{H'} \sim \frac{H'}{H''} \sim \cdots \sim \frac{H^a}{K}$. And $H$ can be generated by $K$ and $a+1$ conjugate groups, one of which is $H^a$.

J, 48* ; J 19, 7 ; N 1, 88 ; N, 94 ; NC, 106 ; H 1, 38 ; W, d, 32 ; B, 127 ; Bi, 50 ; Bu, 220 ; N, a, 337.

Consequently if the last group of a chief-series is composite, the last group is the direct product of simply isomorphic simple groups.

94. Let $r_1$, $r_2$, $r_3$, $\cdots$ be the composition-factors of a group, If $r_i$ and $r_j$ are not equal, a composition-series $G$, $H$, $\cdots$ $I$, $J$. can be arranged so that $\frac{i}{j} = r_j$ and so that $J$ is self-conjugate in $G$. Hence if the composition factors of a group are not all equal, the group has a chief-series. (N, 94 ; NC, 106 ; W, d, 33.)

95. If the composition-factors of a group are all primes, the group is termed *soluble*.

J 1, 192 ; J 5, 111 ; J, 385, 387 ; Pt, 289 ; N, 275 ; NC, 287 ; BD, 304 ; V, 192 ; Pc, 479 ; B, 130 ; Bi, 64 ; Bu, 225 ; H, 497.

96. Subgroups of a soluble group are soluble.

J 5, 119; J, 387; J, 19, 2; N, 277 ; NC, 288; B, 130 ; H, 497.

97. Every abelian group is soluble. (J, 287,—all others.)

98. Every soluble group contains a self-conjugate abelian subgroup.

J 2, 963 ; J 5, 118 ; J, 395 ; Ja 4, 292 ; N 1, 92 ; N, 276 ; NC, 288 ; W, d, 33 ; M 13, 267 ; M 32, 137.

99. If the composition-factors of a soluble group can be taken in any order, the group is the direct product of subgroups whose orders are powers of primes. (B, 131.)

Other properties of soluble groups will be given later. See also B, 131; M 60.

## 7. COMMUTATORS

100. If $S$ and $T$ are two elements, the products $S^{-1}T^{-1}ST$ and $T^{-1}S^{-1}TS$ ($= (S^{-1}T^{-1}ST)^{-1}$) are termed the *commutators* of $S$ and $T$.

F 9, 1348; Dd 1, 553; M 32, 135; Bu, 210; W, d, 133; Al, 141; Sc 3, 376. (Due to Dd.)

101. The commutators of all the pairs of elements of a group $G$ form a self-conjugate subgroup $G_1$ of $G$. $G_1$ is termed the *commutator subgroup* or *first derived group* of $G$.

M 13, 268; F 9, 1348; M 32, 135; M 47, 231, W, d, 134; Sc 3, 376.

Note. If $G$ is abelian, $G_1$ is identity.

102. If $G_1$ and $G$ coincide, $G$ is termed *perfect.* (M 32, 136; M 47, 231.) For a study of such groups see M 36.

103. If a self-conjugate subgroup $H$ of $G$ contains $G_1$, then $\frac{G}{H}$ is abelian. And conversely.

M 13, 267; F 9, 1348; M 32, 137; W, d, 134; Sc 3, 377.* (Due to Dd.)

Note. It follows that a group simply or multiply isomorphic with an abelian group is not perfect. And that $\frac{G}{G_1}$ is the largest abelian group with which $G$ is isomorphic.

104. Every perfect group is isomorphic with some simple group of composite order. (M 36, 278.)

105. Let $G_2$ be the commutator subgroup of $G_1$, termed the second derived group of $G$. And let $G_3$, $G_4$, ⋯ be the third, fourth, ⋯ derived groups of $G$. If one of these groups reduces to identity the group $G$ is soluble. And conversely.

J 2, 963; J 5, 118; J, 395; Ja 4, 291; N 1, 92; N, 276; NC, 288; M 13, 267; M 32, 137; N, a, 335; Sc 3, 378.

106. If the commutators of a group are self conjugate, the group is termed *metabelian*. (Fi 2,—see his articles, especially Fi 5.)

## 8. ABELIAN GROUPS

107. From every abelian group there can be selected a set of generating elements $S_1, S_2, \cdots S_m$, such that the product $S_1^{\alpha} S_2^{\beta} \cdots S_m^{\mu}$ equals every element of the group once and only once as $\alpha, \beta, \cdots \mu$ take all possible values from 0 to $s_1, s_2, \cdots s_m$ respectively. Such a set of elements is termed a *basis* of the group.

K 6, 883; FS, 219; W 1, 306; N, 146; NC, 159; V, 143; W, d, 39; Zs, 211; B, 52 (implicitly); Bi, 76; Bu, 222; Hf 3, 261; N, a, 242.

Note 1. The definition above is that of Weber. Most other writers require that the order of each basis element shall divide the order of the following (a "*reduced*" basis — Hf 3, 261). In order to determine the basis more uniquely other conditions are imposed by Bi and N, a.

Note 2. The basis may be chosen so that the orders of its elements are powers of prime numbers. (W 1, 307; W, d, 45.)

Note 3. For the origin of this concept see Gauss, Works, vol. 2, p. 266 and cf. Sn.

Note 4. If $h$ divides $g$, $G$ contains a subgroup of order $h$.

108. The number of elements in a basis depends upon the choice of the basis. When this number is as small as possible it is termed the *rank* of the group.

FS, 219; Bu, 222; N, a, 249. Cf. F 6, 1030.

109. Let $s_1, s_2, \cdots s_n$; $s_1', s_2', \cdots s_m'$ be the orders of the elements of two distinct bases of a given abelian group. Let $p$ be a prime number and let $s_1 = p^a t_1$, $s_2 = p^b t_2$, $\cdots s_n = p^c t_n$; $s_1' = p^{\alpha} t_1'$, $s_2' = p^{\beta} t_2'$, $\cdots s_m' = p^{\gamma} t_m'$, where these $t$'s and $t'$'s are prime to $p$. Then, excluding the zero-th powers of $p$, the powers $p^a, p^b, \cdots p^c$ are the same as the powers $p^{\alpha}, p^{\beta}, \cdots p^{\gamma}$ ex-

cept as regards order. Then the totality of the different powers of the different primes that divide the orders of the elements of a basis is the same however the basis is chosen. And these powers are termed the *invariants* of the given group.

W, d, 46. Cf. FS, 238; N, 147; NC, 159; W 4, 107; V, 143; Bi, 78; Bu, 222; Hf 3, 261; N, a, 243.

110. Two simply isomorphic abelian groups have the same invariants. And conversely.

W 4, 110; W, d, 48; B, 54; Bi, 78.

111. If $G$ is an abelian group, let $g = 2^{\alpha}p^{\beta}q^{\gamma} \cdots$ where $p$, $q$, $\cdots$ are distinct odd primes. Then $G$ is the direct product of its subgroups of orders $2^{\alpha}$, $p^{\beta}$, $q^{\gamma}$, $\cdots$.

Gauss (l. c.); FS, 231; W 1, 305; W, d, 40; Zs, 211; B, 48; Bi, 70; N, a, 252.

Note. If an abelian group is not divisible it is cyclic and its order is a power of a prime.

112. If an abelian group of order $p^{a}$ has its invariants $p^{m_1}, p^{m_2}, \cdots p^{m_n}$ then each of these is the order of an element of a basis system and $p^{m_1} \cdots p^{m_2} \cdots p^{m_n} = p^{a}$. And the number of types of abelian groups of order $p^{a}$ is equal to the number of possible partitions of $a$ into numbers whose sum is $a$. And a type as, *e. g.*, the above, may be denoted by the symbol $(m_1, m_2, \cdots m_n)$.

Y, 132; Zs, 203; B, 58.

113. Such a group if of type $(1, 1, 1, \cdots 1)$ has

$$\frac{(p^{a}-1)(p^{a-1}-1)\cdots(p^{a-r+1}-1)}{(p-1)(p^{2}-1)\cdots(p^{r}-1)}$$

subgroups of order $p^{r}$. (B, 60; Bg 1, 66.)

Note. For a more complete study of such groups the concept of the characteristics should be introduced. See Dd; W 1; W 3; W 4; W, d, 49–60. Cf. also N, a, 243–256, and (especially) Zs.

## 9. GROUPS WHOSE ORDER IS A POWER OF A PRIME NUMBER

**114. If $G$ is of order $p^m$ it is soluble and all its composition-factors equal $p$.**

**Sy 2, 588; Cp 3, 69, 73; F 2, 282, 285; W, d, 140; B, 64; Bi, 64; Bg 1, 67.**

**115. Each subgroup of index $p$ is self-conjugate. (F 4, 171; F 5, 982; B, 65.)**

**116. $G$ contains self-conjugate elements other than identity. (Sy 2, 588; F 5, 981; B, 62.)**

**117. If $G = p^2$, $G$ is abelian.**

**N, 137; NC, 149; all others.**

**118. If $G = p^m$ and $m > 3$, $G$ contains an abelian subgroup of order $p^3$.**

**M 12, 371; M 13, 233; M 19, 111.**

**119. If $G = p^m$ and $m > p + 4$, then $G$ contains an abelian subgroup of order $p^4$. (M 19, 113.)**

**120. If $G = p^m$ the cogredient subgroup of $G$ is of an order $< p^{m-1}$ unless $G$ is abelian. (B, 63.)**

**121. If $G = p^m$ and $G$ contains $t$ subgroups of order $p^n$ $(n < m)$, then $t \equiv 1 \pmod{p}$.**

**F 5, 990; B, 71; Bg 1, 67.**

**122. If $t = 1$ $G$ is cyclic unless $p = 2$, $n = 1$. (B, 73, 75.)**

**123. If $n = m - 1$ and if the cross-cut of $G$'s $t$ subgroups of order $p^n$ is of order $p^s$, then $t = (p^{m-s} - 1):(p - 1)$. (Ba 3, 508.)**

**124. If $G = p^m$ and if $K$ is a subgroup of $G$ with $k = p^s$ $(s < m - 1)$, then for every integer $n$ $(m > n > s)$ a group $H$ exists of order $p^n$, contained in $G$ and containing $H$. And if $K$ is self-conjugate in $G$, then $H$ can be found so as to be also self-conjugate in $G$.**

**F 4, 173; B 8, 209*; B, 65*.**

**125. Consequently, if $G = p^m$, $G$ has a *chief*-series with factors all equal to $p$. (Bg 1, 67.)**

126. If $G=p^m$ every self-conjugate subgroup of $G$ contains a self-conjugate element of $G$ that is of order $p$. (F 5, 983.)

## 10. SYLOW'S THEOREM AND ITS EXTENSIONS

127. If $p^a$ is the highest power of $p$ that divides $g$, then $G$ contains a subgroup of order $p^a$.

Sy 2, 586; N 4, 249; N, x, 244; Cp 3, 61; N, 49; NC, 53; F 1, 180; F 2, 282; Ml, a, 115; B 3, 10; F 5, 987; W, d, 135; B, 92; Bi, 57; M 38, 325; Bu, 223; An 3.

Note 1. The writers prior to F 1 rested the proof of this theorem on the following lemma:

If $p^f$ is the highest power of $p$ that divides $n!$, then a substitution group can be formed of degree $n$ and order $p^f$.

Ch 3, 196; Ch 5, [352]; S, c, 275; S, e, 302; J, 27; N, x, 238; E, 222.

Note 2. Consequently, by no. 114, $G$ contains subgroups of orders $p^{a-1}, p^{a-2}, \cdots p$ and hence elements of order $p$. For this last see Ch 3, 250; Ch 5, [358]; J, 26; E, 246; M 38, 323.

128. If $G$ contains $k$ subgroups of order $p^a$ then $k\equiv 1 \pmod p$, and these subgroups form a complete conjugate set.

Mt 4, 308*; Sy 1, 109*,114*; Sy 2, 587; Cp 3, 67*; N, x, 246*; N, 132; NC, 145; F 2, 282, 284; Ml, a, 115; B 3, 10; F 4, 174; W, d, 137; B, 92; Bi, 62; M 38, 325; Bu, 223; An 3. For a closer study of $k$ see Sy 3, 215; F 4, 174; M 75; B 20; B 22; etc.

129. If $H$ is the largest subgroup of $G$ that contains a subgroup of order $p^a$ self-conjugately, then $g/h=k$.

Sy 2, 586; N, 132; NC, 145; F 4, 174; W, d, 137; B, 94; Bi, 62; Bu, 223.

130. If $\beta<\alpha$ every subgroup of order $p^\beta$ occurs in a subgroup of order $p^\alpha$.

Cp 3, 68; F 2, 284; B 8, 12; F 4, 174; W, d, 137; B, 94; Bi, 63.

131. If there are $t$ subgroups of order $p^s$ in $G$, then $t \equiv 1 \pmod{p}$. But these subgroups do not necessarily form a complete conjugate set.

F 5, 988; B, 110; Ba 1, 65.

132. If $p$ and $q$ are distinct primes let $G$ contain at least one subgroup of order $p^a q^b$. Then the total number of such subgroups in $G$ is of the form $1 + kp + lq$. (M 38, 326.)

133. Let $G$ contain $m$ subgroups of order $g/p^s$. Then $m \equiv 0$ or $1 \pmod{p}$ according as there are self-conjugate subgroups among them or not. (Ba 1, 60. See for further extensions.)

134. Let $p$ divide $g$. If $n$ can be chosen so that $g > p^{\frac{n(n-1)}{2}}$, then $G$ contains an abelian subgroup of order $p^n$. (M 33, 120.)

## 11. HAMILTONIAN GROUPS

135. The group whose equations are $S_1^4 = S_2^4 = 1$, $S_1^{-1}S_2S_1 = S_2^3$, $S_1^2 = S_2^2$ is termed the *quaternion* group, $Q$. Its order is 8.

Kp, 45; Dd 1, 549; Bu, 223; W, d, 127; Al, 138.

136. $Q$ is simply isomorphic to the group formed by the quaternion unities $1, i, j, k$.

Kp, 45; Dd 1, 551; M 46, 316; W, d, 127.

137. If all subgroups of a non-abelian group are self-conjugate, the group is termed Hamiltonian.

Dd 1, 549; M 42, 1406; M 43, 510; M 47, 229; Bu, 223; W, d, 129; Al, 138.

138. Every Hamiltonian group contains a quaternion subgroup and hence is of even order. (Dd 1, 558; Al, 142.)

139. If an element of a Hamiltonian group is not self-conjugate it is conjugate only to its own powers. And if all the elements of a group have this property the group is Hamiltonian.

Dd 1, 553; W, d, 129; Al, 138.

140. If a Hamiltonian group of order $2^\alpha$ and an abelian group are absolutely permutable, their direct product is a Hamiltonian group.

Dd 1, 560 *; M 43, 514; W, d, 129 *.

141. Conversely, if $G$ is Hamiltonian and $g = 2^\alpha p^\beta q^\gamma \cdots$, where $p$, $q$, $\cdots$ are distinct odd primes, then $G$ contains one and only one subgroup of each of the orders $2^\alpha$, $p^\beta$, $q^\gamma$, $\cdots$, and is the direct product of these subgroups. And these subgroups are all abelian, excepting that of order $2^\alpha$ which is Hamiltonian. (M 42, 1407; M 43, 514.)

142. For a given value of $\alpha$ there is one and only one subgroup of order $2^\alpha$. It contains $2^{\alpha-2} - 1$ (self-conjugate) elements of order 2, and $2^\alpha - 2^{\alpha-2}$ elements of order 4. These last elements are each permutable with half the elements of the group and their second powers are equal. (M 42, 1408 *; M 43, 514.)

143. Such a group contains $2^{2\alpha-6}$ quaternion subgroups. Their cross-cut (of order 2) is the commutator subgroup.

M 42, 1408; M 43, 515; M 46, 319 *.

144. The commutators of a Hamiltonian group are of order 0 or 2. (Dd 1, 557; M 43, 515.)

145. All these commutators occur in the cogredient subgroup. (Dd 1, 553; Al, 143.)

146. The group is the direct product of two abelian subgroups and a quaternion group.

Dd 1, 560; M 46, 318; Bu, 223.

## 12. TRANSITIVITY

147. If by means of the substitutions of a substitution group any mark can be replaced by any other the group is termed *transitive*. Otherwise it is *intransitive*.

Rf, a, 249 ("first class" = intransitive); Ch 4, [294]; Bt 3, 9 *; J 1, 131; S, c, 313; S, e, 340; J, 29; Ja 1, 290; Pt, 256; N, x, 232; Dy 3, 83; N, 68; NC, 70; BD, 260;

V, 83; Pc, 453; W, c, 523; B, 140; Bi, 20; Bu, 212; N, a, 357.

148. The number of substitutions of a transitive group that change a given mark $x_1$ into a certain other mark equals the order of the subgroup formed by all the substitutions that do not affect $x_1$. And the order of this last subgroup equals the order of the group divided by its degree.

Ch 4, [295]; J, 29*; Pt, 257; N, 70; NC, 72; BD, 303; V, 179; B, 140; N, a, 361.

149. A self-conjugate substitution of a transitive $G^n$ is of class $n$ and is regular. (NC, 82; B, 144.)

Note. A self-conjugate subgroup of such a group is evidently of class $n$.

150. Every transitive $G^n$ contains at least $n-1$ substitutions of class $n$. If it contains more it contains also substitutions of a lower class. And conversely.

Ch 5, [393]; J 24, 352; N, 71; NC, 85; F 2, 294; B, 141; N, a, 363.

151. These substitutions of class $n$ form in themselves a transitive system.

N 2, 55; NC, 86; M 9, 77.

152. If two transitive groups of degree $n$ have all their substitutions of class $n$ in common, then they differ only in their substitutions of class $n-1$.

N 2, 52*; N, 72; NC, 87; F 2, 295; M 9, 77.

153. If $G$ is transitive the number of unchanged marks in all its substitutions is $g$. And conversely.

F 2, 287; M 9, 75; B, 165.

Note. In other words, the average class of the substitutions of a group of degree $n$ is $n-1$.

154. If $K$ is the cogredient subgroup of $G^n$, then $k$ equals or divides $n$. (B, 145.)

155. A transitive group of prime degree cannot have more than one composite composition factor. (M 36, 278.)

## 13. INTRANSITIVITY

**156. The marks of an intransitive group can be arranged in sets, in each of which the marks are transitively connected.**

Ch 4, [296]; Ch 10; Mt 1, 1049; J 1, 132; J, 30; Ja 1, 293; Pt, 257; N, 68; NC, 70; Bz 1, 195; W, c, 524; B, 159; Bi, 21.

**157. If these sets contain $a$, $b$, $c$, $\cdots$ $e$ marks respectively (so that $n = a + b + c + \cdots + e$) then $g$ is a common multiple of $a$, $b$, $c$, $\cdots$ $e$, and a divisor of the product $(a!)(b!)(c!)\cdots(e!)$.**

Ch 4, [299]; J, 30; N, 69; NC, 71; B, 159; Bi, 21.

**158. The number of unchanged marks in all the substitutions of an intransitive group is the product of its order by the number of the above sets.**

F 2, 287; M 9, 75; B, 166.

**159. If $G$ and $G'$ are two substitution groups with entirely distinct marks, their direct product is an intransitive group.**

Ch 4, [297]; N, 102; NC, 112; Bz 1, 195; B, 161; M 57, 291.

Note. If $G$ is simple and if $g' < 2g$, then no other intransitive group can be formed from $G$ and $G'$ unless $G \sim G'$. (J 26, 50*; B, 162.)

**160. If an isomorphism of any kind can be established between $G$ and $G'$ the multiplication of every substitution of $G$ by the corresponding one or more of $G'$ will form an intransitive group and to every isomorphism establishable between the two groups there corresponds a distinct intransitive group. This method with that of no. 159 gives all possible intransitive groups derivable from $G$ and $G'$.**

J 26, 48*; N, 102; NC, 112; Bz 1, 200; B, 161; M 57, 291. For a complete discussion see Bz 1.

## 14. PRIMITIVITY

**161.** If the marks of a transitive group can be divided into systems in such a way that each substitution of the group replaces all the marks of every system either by the same marks (in some order) or else by all the marks of some other system the group is said to be *imprimitive.* Otherwise it is *primitive.*

Rf, a, 249 ("second" and "third" "classes"); G 1, 11; G 4, 58; Ch 4, [311]; Bt 2, 68; Bt 3, 9; J 1, 133, 188; Ki 1, 305; J 13, 257; J, 34; Pt, 264; Dy 3, 83; N, 77; NC, 74; BD, 302; V, 181; Pc, 454; W, c, 524; B, 171; Bi, 27; Bu, 212; N, a, 366.

**162.** If three marks are chosen in a primitive group the group contains a substitution replacing one of these by a second and another by some fourth mark. Conversely, a group with these properties is primitive.

Ru, 1; NC, 78; B, 184.

**163.** The subgroup referred to in no. 157 must affect every mark except $x_1$. (Ru, 3.)

**164.** A primitive group cannot have a self-conjugate intransitive subgroup.

Rf 1; J 1, 133 *; J, 41; NC, 82; BD, 302; W, c, 563; B, 187; Bu, 220; N, a, 368.

**165.** If a primitive $G^n$ contains a subgroup of lower degree ($H^m$) then $G$ contains a subgroup of degree $n-1$. (NC, 94.)

**166.** Then on one determined mark of $G$ and $n-1$ other marks arbitrarily chosen a subgroup can be constructed similar to $H$. (Ru, 5; NC, 91.)

**167.** If a primitive group is soluble its degree is a power of a prime.

G 1, 11; Bt 2, 71, 107; J 1, 190; J 5, 125; J 6, 271; J, 398; N, 282; NC, 293; B, 192; Bu, 225.

**168.** If the degree of a primitive group is twice a prime the group is simple. (M 36, 279.)

**169.** If a primitive group contains a transposition it is symmetric.

S, c, 317*; S, e, 344*; N, 80; NC, 91; W, c, 539; Bi, 28.

**170.** If a primitive group contains a circular substitution of order three the group is alternating or symmetric.

S, c, 317*; S, e, 344*; N, 80; NC, 91; H 4, 63*; W, c, 541; B, 152*.

**171.** If a primitive group of degree $n$ contains a primitive subgroup of degree $m$ then $m \nless \frac{2}{3} n$.

N 2, 48*; Ru, 7*; NC, 93*; B, 199.

**172.** If a primitive group of degree $n$ contains an imprimitive subgroup of degree $m$ then $m \nless \frac{2}{3} n$ unless $n = 2^a$. Then $m \nless \frac{1}{2} n$. (Mg.)

**173.** If a primitive group of degree $n$ contains a transitive subgroup of prime degree $p$, then $n = p$, $p + 1$, or $p + 2$. (M 34, 141.)

**174.** If $G$, of degree $n$, is not alternating or symmetric, and if $2, 3, 5, \cdots p$ are the distinct primes less than $\frac{2}{3} n$, then $g \ngtr \frac{n!}{2 \cdot 3 \cdot 5 \cdots p.}$ (B, 199.)

**175.** The number of primitive groups of a given class is limited (J 21, 408) as is the order of a primitive group of given degree. See J 21; J 24; J 28; J 29; J 30; N 5, 334; Ml, 9. For primitive groups of class $n - 1$ see Ml, a, 49 ff; Ml 11; Ml 16. For class $n - 2$ see Ml, a, 70 ff; Ml 18. For other classes see Ml, a, 102 ff; Ml 14; Ml 20. (In these multiple transitivity is introduced—cf. no. 210.)

## 15. REGULAR GROUPS

**176.** If the degree and order of a transitive group are equal the group is termed *regular*.

Dy 3, 86; NC, 110; Bz, 81; B, 24; H, 505. Used by all others.

**177.** Every substitution of a regular group of degree $n$ is regular and of class $n$.

J, 58; N, 99; NC, 110; F 4, 180; V, 180; B, 24.

178. A regular group contains one and only one substitution replacing a given element by a prescribed element.

J, 58; N, 99; NC, 110; V, 180; B, 23.

179. A regular group of composite order is imprimitive.

Ft 1, 165; NC, 110; F 4, 180; B, 173.

180. Two simply isomorphic regular groups are similar.

J, 59; N, 99; NC, 110; M 30, 239; M 35, 68.

181. Corresponding to every regular group $R$ there is a second $R'$, simply isomorphic to $R$ and absolutely permutable with it. And any substitution on $R$'s marks that is permutable with all its substitutions is in $R'$.

J 1, 153*; J, 60; Ft 1, 160; Ml 1, 124 (with an extension); B, 146; M 30, 235.

182. *dv* $(R, R')$ is the cogredient subgroup of $R$ and also of $R'$. If $R$ and $R'$ are abelian they coincide. (Ft 1, 153; B, 146.)

183. If $R$ and $R'$ are simple then $\{R, R'\}$ is primitive, and has composition-factors $r$, $r$. (Ml 1, 131.)

184. A transitive group of degree $n$ and class $n-1$ contains a regular subgroup of degree $n$.

B 17, 240*; F 17, 1226. Cf. F 20 for an extension.

## 16. IMPRIMITIVITY

185. If the marks of an imprimitive group can be divided into systems in two distinct ways, then a third division is obtained by taking from each system of one division the marks that are found in one system of the second division.

J, 34; N, 77; NC, 75; B, 185.

Note. J, 34 adds that a fourth division can be found by uniting the systems of the first division that have marks in common with one of the second division, and from this develops a concept "factors of imprimitivity." But the discussion (reproduced in Ja 1) is not general as is shown in J 23 and very elaborately in Ft 2. In the latter paper the conditions for its truth are fully developed. Cf. Wk; B, 186; Bu, 212; Lm 1, 138.

186. Every imprimitive group contains at least one substitution that effects a simultaneous interchange of all the systems of imprimitivity. (N 2, 55; NC, 86.)

187. If an imprimitive group $G$ has a self-conjugate intransitive subgroup $H$, then the systems of intransitivity of $H$ are the systems of imprimitivity of $G$.

NC, 82; W, c, 563; B, 187.

188. If an imprimitive group $G$ has $\mu$ systems of imprimitivity, on these systems as marks a transitive substitution group $G'$ can be constructed that is isomorphic to $G$.

Pt, 267; Dy 3, 90; NC, 82; F 4, 179; B, 187; Lm 1, 185.

Note 1. If $G$ contains substitutions that interchange elements within the systems but not the systems this isomorphism is multiple, $g' < g$, and $G$ is composite. Otherwise $g' = g$. (Dy 3, 94; Lm 1, 135. N, 94 is incorrect.)

Note 2. If $G$ is perfect $G'$ is perfect. (M 36, 278.)

## 17. MULTIPLE TRANSITIVITY

189. If by means of the substitutions of a substitution group any set of $t$ marks can be replaced by any set of $t$ prescribed marks the group is termed *t-fold* transitive.

Ch 7, [413 *]; Mt, 16; J 1, 131; S, c, 313; S, e, 340; J, 29; Pt, 257; N, 72; NC, 72; Dy 3, 84; B, 148; Bi, 22; Bu, 214; N, a, 361.

Note. $t$ is termed the *degree of transitivity*.

190. The symmetric group of degree $n$ is $n$-fold transitive, the alternating group is $(n-2)$-fold transitive.

S, c, 314; S, e, 341 (error in both for symmetric group); J, 66 (error for alternating group); N, 73; NC, 73; F 2, 292; B, 153; Bi, 22.

191. If $G$ is $t$-fold transitive then $g = n(n-1)\cdots(n-t+1)\mu$, where $\mu$ is the order of the subgroup formed by the substitutions of $G$ that leave $t$ prescribed marks unaffected. And $G$ contains substitutions of classes $n, n-1, \cdots n-t+1$.

S, c, 315; S, e, 341; Mt 3, 14; J 29 *; Pt, 258; N, 74; NC, 74; Cs 1, 472; F 2, 290; B, 148; Bi, 22; N, a, 362*.

192. Such a subgroup of order $\mu$ is self-conjugate in a subgroup of order $t! \cdot \mu$. (B, 148.)

193. $G$ contains at least $\frac{g}{\mu} - 1$ substitutions of a class greater than $n - t$. (Cs 1, 474.)

194. If $\mu > 1$ $G$ contains $\mu - 1$ substitutions of a class not greater than $n - t$. (B, 149.)

195. If $\mu = 1$ $G$ is of class $n - t + 1$ and there is only one substitution in $G$ that will replace a given set of $t$ marks by a prescribed set. (B, 149.)

196. Let $G$ be known to be at least $r$-fold transitive. If it contains an $s$-fold transitive subgroup of degree $n - r$, then $G$ is at least $(r + s)$-fold transitive. (Pt, 265; NC, 87; F 2, 290.)

Note. A method to determine whether a given $t$-fold transitive group of order $g$ and degree $n$ is contained in a $(t + 1)$-fold transitive group of order $(n + 1)\ g$ and degree $n + 1$ is given in J, 30; B, 150.

197. If a primitive group $G$ contains a transitive subgroup $H$ of degree $m$ then $G$ is at least doubly transitive.

N, 84; NC, 95; B, 198.

198. If $H$ in the above is primitive then $G$ is at least $(n - m + 1)$-fold transitive.

N 2, 48; N, 84 *; Ru, 6; NC, 92; B, 197.

199. If $G$ is doubly transitive and $g = n(n - 1)$, then $n$ is a power of a prime. (J 24, 353.)

200. If $G$ is $t$-fold transitive $(2 < t < n)$, a self-conjugate subgroup is in general at least $(t - 1)$-fold transitive. But a triply transitive group of degree $2^m$ may have a (simply transitive) self-conjugate subgroup of degree and order $2^m$.

J, 65 *; J 26, 69, 70; Pt, 261; B, 189.

201. If $t > 1$ $G$ is simple or contains a simple self-conjugate subgroup. To this there is an exception as above and

also for a doubly transitive group whose degree is a power $p^m$ of a prime, such a group having possibly a (composite) self-conjugate subgroup of order $p^m$. (B, 192.)

202. If two groups of the same degree $n$ and more than doubly transitive have all their substitutions of class $n$ in common, then they are identical.

N 2, 53; Cs 1, 475*; F 2, 295.

## 18. CLASS OF A GROUP AND DEGREE OF TRANSITIVITY

Note. In what follows, $n =$ the degree, $u =$ the class, $t =$ the degree of transitivity of the group. And the group is neither alternating nor symmetric.

203. $u > t$.

Mt 3, 17; J, 64; Pt, 258; F 2, 297; Bi, 23.

204. $u > 2t - 3$.

J, 64*; Pt, 259*; N, 75*; Bo 2, 33; NC, 88; F 2, 298; B, 152.

205. If $t > 1$, $u > \frac{1}{4}n - 1$. If $t > 2$, $u > \frac{1}{3}n - 1$. If $t > 3$, $u > \frac{1}{2}n - 1$. (Bo 6, 179.)

206. If $t > 1$ then $u$ is greater than the largest of $\frac{1}{6}n$, $\sqrt{n}$, $\frac{1}{3}n - \sqrt{n}$. And if $u > 3$ then $u > \frac{1}{3}n - \frac{2}{3}\sqrt{n}$. (Bo 8, 143.)

207. If $p$ is any prime and if $n - p = k$ $(k > 2)$, then $t \not> k$.

J, 86*; J 26, 42; J 28, 953; N 2, 50; M 34, 142.

208. Or if $n - 2p = k'$ $(k' > 2)$, then $t \not> k'$. (J 26, 43*; M 34, 143.)

209. Or, if $q$ is any number prime to $p$ let $n - p^m q = l$. Then $t \not> l$ unless (*a*) $l < 5$,—or (*b*) $l \not> q$,—or (*c*) a very special case.* (J 26, 50; J 28, 953.)

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* If $G$ is a subgroup of the linear group of degree $p^a$ and if $G$ contains a factor group simply isomorphic to the alternating group of degree $t$, then $t - a \not> \frac{\log t}{\log 2} + 3$.

210. This subject has been extensively studied and very close limits have been determined but the results are often too intricate for any but the special student. See J 9; J, 29–37; J 21; J 28; J 29; J 30; all of Bo; F 2, 296 seq.; Ru; Mg; J 32; dS 3.

211. The closest limit for $t$ in groups of high degree is that of J 32, 37, viz.: $\log(n - t) \ngtr (1 + e) \log 2 \sqrt{t \log t}$, where $e$ approaches zero as $t$ increases. Here $n > 7$.

212. There exists a group with $n = 12$, $t = 5$, and in this group the ratio $t : n$ is a maximum.

Mt 4, 270; J, 33; NC, 89; the group lists.

## 19. AUTOMORPHISM

213. Let the elements of a group be arranged in two orders. If these two arrangements are such that in them corresponding pairs of elements have the same law of combination they are said to define an *isomorphism of the group with itself* or, more briefly, an *automorphism*. Each automorphism defines a substitution on the elements of group as marks and the totality of such substitutions form the *group of automorphisms* (or *i-group*) of the given group.

H 5, 314 and Mo 3, 61 (independently); F 4, 184; B 12, 355; B, 223; M 30, 235; Bu, 221; F 18, 1324 ("automorphism").

Note. The concept "automorphism" is used explicitly in Gs, 354; Kl, 232.

214. If all the elements of a group are transformed in order by one among them the resulting automorphisms are termed *cogredient*. Other automorphisms are *contragredient*.

Gs, 354; Kl, 227, 232; H 5, 314; B 12, 356; B, 224; M 30, 236; Bu, 220; F 18, 1324 ("inner" and "outer").

215. The cogredient automorphisms form a self-conjugate subgroup of the whole *i*-group, the co-*i*-group (say).

H 5, 314; H 6, 326; B 12, 356; B, 224; M 30, 236; Bu, 220.

216. The co-*i*-group of a direct product is the direct product of the co-*i*-groups of the factors. (M 48, 294.) And, conversely, if the co-*i*-group is the direct product of two subgroups of relatively prime order, then the group is the direct product of two subgroups. (M 60, 69.)

217. If $H$ is the cogredient subgroup of $G$ then the co-*i*-group of $G$ is simply isomorphic to $G/H$.

B 12, 357 ; B, 225 ; Bu, 220.

218. Automorphisms that are found in the same adjunct group to the co-*i*-group are said to belong to the same *class*.

H 6, 326 ; B, 224 ; M 30, 237.

219. The direct product of $G$ and its *i*-group (represented abstractly) is termed the *holomorph* of $G$.

F 4, 185 ; B, 228 ; Bu, 221.

Note 1. Every automorphism of $G$ can be obtained by transforming $G$ by an element of the holomorph. (F 4, 185.)

Note 2. Bu, 221, note 103 is not accurate.

Note 3. The automorphism $\Sigma^{-1}G\Sigma$ may be represented by $(\Sigma)$. If $\Sigma$ is in the group $\Gamma$ then $(\Sigma)$ is said to be in $\Gamma$. (F 18, 1324.)

220. A subgroup of $G$ that corresponds to itself in every automorphism of $G$ is termed *characteristic*.

F 4, 183 ; B 12, 357 ; B, 232 ; M 30, 236 ; Bu, 221.

Note. Such a subgroup is self-conjugate in the holomorph.

221. A group that has no characteristic subgroup either is simple or is the direct product of simply isomorphic simple groups. (B 12, 360 ; B, 232.)

222. If a series of characteristic subgroups of a group is formed, analogous to a chief-composition series, this series is termed *characteristic*, and its factor groups are invariable (except as regards order) however the series is taken.

F 6, 1027 ; B, 232 ; Bu, 221.

Note 1. For the formation of such a series when $g = p^a$ ($p =$ a prime) see F 6, 1028 ; B, 233.

Note 2. Any self-conjugate subgroup of a unique type is characteristic. *E. g.*, the cogredient subgroup, the commutator subgroup, etc.

223. A group that contains no self-conjugate element (except 1) and that admits of no cogredient automorphism is termed *complete*.

H 6, 325; B 12, 363; B, 236; M 30, 242; Bu, 221.

Note. The holomorph of a complete $G$ is of order $g^2$.

224. If a group $G$ contains a complete group $H$ self-conjugately then $G$ is the direct product of $H$ and a subgroup simply isomorphic to $G/H$.

H 6, 325; B, 236; Bu, 221.

225. The symmetric substitution-group of degree $n$ is complete unless $n = 6$. (H 6, 345; B, 246.)

226. An automorphism that is defined by a correspondence between each element of a group and the $a$-th power of the same element is termed an *a-holomorphism*. If a group admits such a holomorphism then the $(a-1)$-st power of every element is self-conjugate and an $(a-1)$-holomorphism is also possible. (Yn 1, 186, 187.)

227. The automorphisms of a substitution-group present a slightly greater difficulty than those of an abstract group, inasmuch as no. 219, note 1 does not hold. For a discussion see M 24; M 30; M 31.

## 20. REPRESENTATION

Note. Maillet (Ml, a, 31; Ml 9, 85) has introduced the term *isomorph* to describe a group isomorphic to a given group. (The isomorphism is simple unless otherwise stated.) A substitution group that is an isomorph of a given abstract group is said to *represent* the group.

228. Every group $G$ (abstract or otherwise) has a regular simply isomorphic isomorph $R$.

J, 57; Cp 3, 45; Dy 2, 30; Dy 3, 85; N, 99; NC, 110; Ft 1, 144; BD, 260; F 4, 180; M 13, 265; B, 22.

229. Corresponding to every subgroup $H$ of $G$ there is a division of the $g$ marks of $R$ into $g/h$ systems of imprimitivity. And conversely.

Dy 3, 89; Ft 1, 148; Ml, a, 10; B, 173.

230. If $H$ is not self-conjugate and contains no self-conjugate subgroup of $G$, then $G$ has an isomorph, transitive and of degree $g/h$. And, conversely, if $G$ has such an isomorph it contains such a subgroup.

Dy 3, 91; Ml, a, 12; F 4, 180; B, 175.

231. If this subgroup $H$ is contained in a larger subgroup $H'$ then this isomorph is imprimitive and contains $g/h'$ systems of imprimitivity. And conversely.

Dy 3, 94; Ml, a, 13; B, 176; M 35, 67.

232. The number of distinct transitive isomorphs of a given group $G$ equals the number of systems of subgroups of $G$ that satisfy the two following conditions:

1. Each system includes all subgroups that can be brought in correspondence in any possible automorphism of $G$.
2. A subgroup that belongs to such a system includes no elements generating any self-conjugate subgroup.

All these isomorphs can be obtained directly from the given $G$. (M 24, 215; M 35, 65.)

233. If a divisible group $G$ has a primitive isomorph, then $G$ is the product of but two factors, which are simply isomorphic simple groups of composite order. And conversely. The representation is unique and is of degree $\sqrt{g}$.

Ml, a, 31; Ml 1, 131; B, 190; M 60, 70.

234. For multiply transitive isomorphs see M 51, for intransitive isomorphs see F 4, 183. For multiply isomorphic isomorphs see Ml, a, 14; F 4, 179; M 36, 280; M 41, 86.

235. A convenient method of obtaining an isomorph is by transforming all the elements or subgroups of a complete set by all the elements of the group in order. The isomorph obtained will be simply isomorphic when and only

when the group contains no self-conjugate element. This method reduces in theory to one of the above.

CoG, 195; F 4, 182; B, 178; Cf. D 29; D 30.

## 21. THE INDEX NOTATION

236. A substitution may be represented analytically by a formula defining the relation of the final value of the index of each mark to its initial value.

Ch 3, 232; Bt 1, 8; Ht 3, 750; Mt 5, 115; all later writers.

237. If $|s, \varphi(s)|$ represents a substitution, then $\varphi(s) = 1, 2, \cdots n$, in some order as $s = 1, 2, \cdots n$. (Ch 13, [466].)

238. A formula can be found to represent any given substitution — Lagrange's interpolation formula, *e. g.* But the determination of *all* such formulæ has not been accomplished. Cf. D, 54–71. But:

239. The necessary and sufficient conditions that $\varphi(s)$ shall represent a substitution of degree $p^a$ are:

1. If $t < p^a - 1$ and is prime to $p$, then every $t$-th power of $\varphi(s)$ shall reduce to a degree $< p^a - 1$ on applying the congruence $s^{p^a} \equiv s \pmod{p^a}$.

2. There shall be one and only one root of $\varphi(s) \equiv 0 \pmod{p^a}$.

D 2, 73; D, 59. For $a = 1$ in Ht 3, 751; S, c, 358; S, e, 385; J, 89; N, 148; NC, 161; BD, 306. For various special forms see Bsc; dP; Gr 1; Gr 2; Rg 2.

240. For $n = 5$, see Ht 3, 752; Rg 2, 45 and the above, for $n = 6$ see Bt 5, for $n = 7$ see Ht 3, 753; Bsc, Rg 1; Rg 2, 46, more generally in D 1, 218; D 2, 119; D, 63.

# TABLES

## 1. DISTINCT ABSTRACT GROUPS

| Order: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number: | 1 | 1 | 2 | 1 | 2 | 1 | 5 | 2 | 2 | 1 | 5 | 1 | 2 | 1 |

| Order: | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number: | 14 | 1 | 5 | 1 | 5 | 2 | 2 | 1 | 15 | 2 | 2 | 5 | 4 | 1 |

| Order: | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number: | 4 | 1 | 51 | 1 | 2 | 1 | 14 | 1 | 2 | 2 | 14 | 1 | 6 | 1. |

| Order: | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number: | 4 | 2 | 2 | 1 | 52 | 2 | 5 | 1 | 5 | 1 | 15 | 2 | 13 | 2 |

| Order: | 58 | 59 | 60 | 61 | 62 | 63 | 168 |
|---|---|---|---|---|---|---|---|
| Number: | 2 | 1 | 13 | 1 | 2 | 4 | 57. |

1–35 in M 12, 36–47 in M 13, 48–63 in M 44, M 45. 168 in M 86; M 91.

For 16 see M 12, for 32 see Bg 3. Lists 1–12 in Kp, 37–43; C 18. Earliest lists in C 1; C 3.

## 2. DISTINCT SUBSTITUTION GROUPS

| Degree: | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| Multiply transitive: | 1 | 2 | 3 | 4 | 4 | 7 | 9 Co 7 | 7 Co 9 |
| Other primitive: | 1 | 0 | 2 | 0 | 3 | 0 | 2 Co 7 | 2 Co 9 |
| Imprimitive: | 0 | 3 | 0 | 12 | 0 | 43 M 57 | 23 Co 7 | 36 M 4 |
| Intransitive: | 0 | 2 | 3 | 21 | 33 | 150 | 224 M 2 | 994 M 3 |
| Total: | 2 | 7 | 8 | 37 | 40 | 200 | 258 | 1039 |

| Degree: | 11 | 12 | 13 | 14 |
|---|---|---|---|---|
| Multiply transitive: | 5 Co 9 | 6 M 8 | 4 M 26 | 4 M 26 |
| Other primitive: | 3 Co 9 | 0 M 8 | 5 M 26 | 0 M 26 |
| Imprimitive: | 0 — | 298 M 26 | 0 — | 59 M 26 |
| Intransitive: | 1492 ML 2 | | | |
| Total: | 1500 | | | |

| **Degree:** | **15** | **16** | **17** | **18** |
|---|---|---|---|---|
| **Multiply transitive:** | **4 M 27** | **15 M 28** | **6 M 55** | **4 Mrt** |
| **Other primitive:** | **2 M 27** | **7 M 28** | **4 M 55** | **0 Mrt** |
| **Imprimitive:** | **98 Ku** | | **0** | |

### Other References:

**3. Ch 15; Ki 5; Vr; As 1; C 19.**
**4. Ch 15; S 3; Ck; Ki 5; Vr; As 1; C 19.**
**5. Ch 15; S 3; Ck; Ki 5; Vr; As 1; C 19.**
**6. Ch 15; Ki 5; Vr; As 1; C 19; Co 5; Co 7.**
**7. Mt 1; Ki 5; As 1; C 19; Co 5; Co 7.**
**8. Mt 1; Ki 5; As 1; C 19; Co 5; Co 7; M 1; M 2.**
**9. Ki 5; As 3 (tr.); Co 6 (tr.); M 2.**
**10. Ki 5; Co 9 (tr.); M 52 (pr.).**
**11. J 25 (pr.).**
**12. J 25 (pr.); M 8.**
**13. J 25 (pr.).**
**14. J 25 (pr.).**
**15. J 25 (pr.); Mrt (impr.).**
**16. J 25 (pr.).**
**17. J 25 (pr.).**
**19. J 31 (tr.).**

### 3. Soluble Primitive Groups

| **Degree:** | **3** | **4** | **5** | **7** | **8** | **9** | **11** | **13** | **16** | **17** | **19** | **23** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Number:** | **2** | **2** | **3** | **4** | **2** | **7** | **4** | **6** | **10** | **5** | **6** | **4** |

**See M 28.**

### 4. Non-soluble Abstract Groups

| **Order:** | **60,** | **120,** | **168,** | **180,** | **240,** | **300,** | **336,** | **360,** | **420.** |
|---|---|---|---|---|---|---|---|---|---|
| **Number:** | **1,** | **3,** | **1,** | **1,** | **8,** | **1,** | **3,** | **6,** | **1.** |

**No others of order < 480. See H 6, 420; B, 376.**

### 5. Primitive Groups of Odd Order

**If the self-conjugate subgroups of the metacyclic group are excepted, the only such groups of degree < 243 are those**

of orders $25 \cdot 3$, $27 \cdot 13$, $27 \cdot 39$, $81 \cdot 5$, $121 \cdot 3$, $121 \cdot 15$, $125 \cdot 31$ $125 \cdot 31 \cdot 3$, $169 \cdot 7$, $169 \cdot 21$. Their degrees are the first factors of their orders. (Rz 1; Rz 2.)

## 6. TYPES OF GROUPS OF GIVEN ORDER

The following table gives on the left the abstract groups of each type and on the right the *additional* transitive substitution groups. In the substitution groups $n$ is the degree.

(a). $g = p^2$.

| | |
|---|---|
| 2 types, both abelian. | None. |
| N, 136; NC, 148; Y; CoG, 192; H5, 302. | |

(b). $g = pq (p < q)$.

| | |
|---|---|
| 1 abelian. | None. |
| 1 non-abelian if $p$ divides $q-1$. | 1 if $p$ divides $q-1$. |
| N, 133; NC, 146; CoG, 193; H 5, 303. | M 16; M 18. |

(c). $g = p^3$.

| | |
|---|---|
| 3 abelian. | None. |
| 2 non-abelian. | 1 if $p = 2$, 2 if $p > 2$. |
| Y; CoG, 196; H 5, 371; B, 87; Bg, 2, 158. | M 15; M 24. |

(d). $g = pq^2$.

| | |
|---|---|
| 2 abelian. | None. |
| Non-abelians: | |
| 3 if $p = 2$. | 3 with $n = q^2$, 1 with $n = pq$. |
| $\frac{1}{2}(p+5)$ if $p > 2$ and divides $q-1$. | $\frac{1}{2}(p+5)$ with $n = q^2$, $\frac{1}{2}(p+1)$ with $n = pq$. |
| 3 if $g = 12$. | 1 with $n = 4$, 1 with $n = 6$. |
| 1 if $p > 2$ and divides $q+1$. | 1 with $n = q^2$, 1 with $n = pq$. |
| 2 if $p > 3$, $p-1$ is divided by $q$ but not $q^2$. | 1 with $n = pq$. |

| | |
|---|---|
| 3 if $p-1$ is divided by $q^2$. | 2 with $n = pq$, 1 with $n = p$. |
| CoG, 200, 209; H 5, 335. | |

### (e). $g = pqr$ $(p > q > r)$.

| | |
|---|---|
| 1 abelian. | None. |
| Non-abelians: | |
| (α). $r$ divides $q-1$: | |
| $r+3$ if $q$ and $r$ divide $p-1$. | 1 with $n = p$, 2 with $n = pr$, $r+2$ with $n = pq$. |
| 2 if $q$ divides, $r$ does not divide $p-1$. | 1 with $n = pr$, 1 with $n = pq$. |
| $r+1$ if $r$ divides, $q$ does not divide $p-1$. | $r+1$ with $n = pq$. |
| 1 if neither $r$ nor $q$ divide $p-1$. | 1 with $n = pq$. |
| (β). $r$ does not divide $q-1$: | |
| 3 if $q$ and $r$ divide $p-1$. | 1 with $n = p$, 2 with $n = pr$, 2 with $n = pq$. |
| 1 if $q$ divides, $r$ does not divide $p-1$. | 1 with $n = pr$. |
| 1 if $r$ divides, $q$ does not divide $p-1$. | 1 with $n = pq$. |
| CoG, 215; H 5, 361. | M 21; M 24. |

### (f). $g = 8p$ $(p > 2)$.

| | |
|---|---|
| 3 abelians. | None. |
| Non-abelians: | |
| 12 if 8 divides $p-1$. | 1 with $n = p$, 2 with $n = 2p$, 9 with $n = 4p$. |
| 11 if 4 divides, 8 does not divide $p-1$. | 1 with $n = 2p$, 8 with $n = 4p$. |
| 9 if 4 does not divide $p-1$. | 6 with $n = 4p$. |
| 12 if $g = 24$. | 1 with $n = 4$, 3 with $n = 6$, 3 with $n = 8$, 10 with $n = 12$. |

| | |
|---|---|
| **10 if $g = 56$.** | **1 with $n = 8$, 1 with $n = 14$, 7 with $n = 28$.** |
| **M 17; M 18; Ws.** | **M 23.** |

**(*g*). $g = 16$.**

| | |
|---|---|
| **5 abelian.** | **None.** |
| **9 non-abelian.** | **6 with $n = 8$.** |
| **Y; H 5, 371; B, 87; Bg 2, 168.** | **M 23.** |

**The following tables give only abstract groups:**

**(*h*). $g = p^4$ $(p > 2)$.**

**5 abelian, 10 non-abelian types. Y; H 5, 371; B, 87; Bg 2, 168.**

**(*i*). $g = p^3q$.**

**3 abelian.**

**Non-abelian:**

**12 if $q = 2$ (M 44).**

**$q > 2$.**

**(α). $p + 6$ if $p$ divides, $p^2$ does not divide $q - 1$.**
**$p + 8$ if $p^2$ divides, $p^3$ does not divide $q - 1$.**
**$p + 9$ if $p^3$ divides $q - 1$.**

**(β).**
$$\left.\begin{array}{l}\frac{1}{6}\,(q^2 + 13\,q + 48)\\ \frac{1}{6}\,(q^2 + 13\,q + 52)\end{array}\right\}\begin{array}{c}\text{if } q \text{ divides } p - 1\\ \text{and 3 divides}\end{array}\left\{\begin{array}{l}q \text{ or } q + 1.\\ q - 1.\end{array}\right.$$
**4 if $q$ divides $p + 1$.**
**3 if $q$ $(> 3)$ divides $p^2 + p + 1$.**

**(γ). 2 in all other cases.**

**All in Ws.**

**(*j*). $g = 32$.**

**7 abelian, 44 non-abelian. M 12; M 45; Bg 3.**

**(*k*). $g = p^5$ $(p > 2)$.**

**7 abelian.**

Non-abelian:

59 if $p = 3$
$2p + 58$ if 12 divides $p + 1$.
$2p + 60$ if 12 divides $p - 5$.
$2p + 62$ if 12 divides $p + 5$.
$2p + 64$ if 12 divides $p - 1$.

All in Bg 2.

## 7. Simple Groups

All groups of prime order are simple. Otherwise if $g < 2001$, then 60, 168, 360, 504, 660, and 1092 are the only orders for which simple groups exist.

G 5, 26 (1–60); H 4 (1–200); Co 3 (201–500); Co 4 (do.–660); B 9 (661–1092); ML 1 (1092–2000).

No simple group of odd composite order is known. If such a group exists its order is not less than 40,000 (B 20), its degree (when represented as a substitution group) is not less than 243 (Rz 2, cf. M 75; B 19).

If the order of a simple group is even that order is divisible by 12, 16, or 56. (B 9, 332; B, 365.)

## 8. Composite and Soluble Groups

If the order of a group has any of the following values the group is composite. And unless the order is marked with an asterisk the group is soluble. $p$, $q$, $r$, denote distinct primes, $p_1$, $p_2$, .. distinct primes in ascending order of magnitude.

$pq$ or $pqr$. (H 4, 61.)

$g$ contains no quadratic factor. (F 3, 342; B 8, 196; W, d, 140; B, 353. Cf. H 7; F 6, 1043.)

* $g$ contains less than six prime factors ($g = 60$, 168, 660, and 1092 excepted). (B 8, 214; F 6, 1042; B, 367.)

$p^a q$. (F 3, 340; F 4, 185; B 8, 210; B, 348; W, d, 145.)

$p^a q^2$. (B 8, 210*; B, 348; J 33.)

* $p^2 q^2 r^2$. (Soluble except for certain small values of $q$.) (Ml 15, 263.)

$p^{\alpha}q^{\beta}$, where if $p$ appertains to $m$ (mod $q$) then $\alpha < 2m$. (F 3, 344*; F 4, 190*; B 8, 209*; B, 345.)

$p^4qr$. (F 18, 1329.)

$p^3n$, where $n$ is prime to $p$, $p^2 - 1$.* (F 17, 1221.)

* If $p$ divides $g$ and $s$ is the greatest factor of $g$ that is $\equiv 1$ (mod $p$) while $s!$ is not divisible by $g$. (H 4, 57; B, 344.)

$p_1^3p_2^{\alpha}$, $p_1^4p_2^{\alpha}$, $p_1^5p_2^{\alpha}$. (F 3, 344*; B, 348.)

$p_1^2p_2p_3^{\alpha}$, unless $g = 2^2 \cdot 3 \cdot 5^{\alpha}$. And $G$ is composite then. (F 6, 1042.)

$mp^{\alpha}q$ where $m$ has no quadratic factor and $m\varphi(m)$ is prime to $pq$. (F 4, 192.)

$p_1p_2 \cdots p_{n-1}p_n^{\alpha}$ or $p_1p_2 \cdots p_{n-2}p_{n-1}^2p_n^{\alpha}$. (F 3, 344.)

$*p_1^2p_2 \cdots p_n$, $*p_1^2p_2 \cdots p_{n-2}p_{n-1}^2p_n$ (if $p_2 \neq 3$) or $*p_1p_2^mp_3$. (B 8, 206–210.)

### 9. SYSTEMS OF SIMPLE GROUPS

The earliest of such systems is the so-called "generalized modular group" discovered by Mt 3, 38–40 and rediscovered by Mo 2 and B 5; B 7. It is included in 1 and 2 below.

1. LF $(m, p^n)$. $g = 1/d\,(p^{nm} - 1)\,p^{n(m-1)}\,(p^{n(m-1)} - 1)\,p^{n(m-2)} \cdots (p^{2n} - 1)\,p^n$,—where $p^n > 3$ if $m = 2$ and where $d = dv\,(m, p^n - 1)$. (D 2, 175; D, 87.)

2. $HO\,(m, p^{2n})$. $g = 1/g'\,[p^{nm} - (-1)^m]\,p^{n(m-1)}\,[p^{n(m-1)} - (-1)^{m-1}]\,p^{n(m-2)} \cdots [p^{2n} - 1]\,p^n$, —where $p^n > 3$ if $m = 2$ or $p^n > 2$ if $m = 3$, and where $g' = dv\,(m, p^n + 1)$. (D 13, 574; D, 134, 138.)

3. $A(2m, p^n)$. $g = 1/a\,(p^{n(2m)} - 1)\,p^{n(2m-1)}\,(p^{n(2m-2)} - 1)\,p^{n(2m-3)} \cdots (p^{2n} - 1)\,p^n$, —where $p^n > 3$ if $m = 1$ or $p^n > 2$ if $m = 2$, and where $a = 1$ if $p = 2$ or $a = 2$ if $p > 2$. (D 3, 173; D, 100.)

4. $FO(2m + 1, p^n)$. $g = \frac{1}{2}(p^{n(2m)} - 1)\,p^{n(2m-1)}\,(p^{n(2m-2)} - 1)\,p^{n(2m-3)} \cdots (p^{2n} - 1)\,p^n$—where $p > 2$ and, for $m = 1$, $p^n > 3$. (D 7*; D 9*; D 11, 256; D, 160.)

---

* If $G$ contains a linear subgroup of order $p^3$ then $n$ must be prime to $p^2 + p + 1$ as well.

5. $FO(2m, p^n)$. $g = \frac{1}{2}[p^{n(2m-1)} - \epsilon^m p^{n(m-1)}](p^{n(2m-2)} - 1) p^{n(2m-2)} \cdots (p^{2n} - 1) p^n$, — where $p > 2$ and $m > 2$, while $\epsilon = \pm 1$ according as $p^n = 4l \pm 1$. (As above.)

6. $SO(2m, p^n)$. $g = \frac{1}{2}[p^{n(2m-1)} + \epsilon^m p^{n(m-1)}](p^{n(2m-2)} - 1) p^{n(2m-2)} \cdots (p^{2n} - 1) p^n$, — where $p > 2$ and $m > 1$, while $\epsilon = \pm 1$ according as $p^n = 4l \pm 1$. (As above.)

7. $FH(2m, 2^n)$. $g = (2^{nm} - 1)(2^{2n(m-1)} - 1) 2^{2n(m-1)} (2^{2n(m-2)} - 1) 2^{2n(m-2)} \cdots (2^{2n} - 1) 2^{2n}$, — where $m > 2$. (D 4, 265*; D 5, 2; D 8, 500; D, 206.)

8. $SH(2m, 2^n)$. $g = (2^{nm} + 1)(2^{2n(m-1)} - 1) 2^{2n(m-1)} (2^{2n(m-2)} - 1) 2^{2n(m-2)} \cdots (2^{2n} - 1) 2^{2n}$, — where $m > 1$. (D 4, 265*; D 8, 506*; D 11, 256; D, 206.)

9. $H'(p^{2n})$. $g = p^{6n}(p^{6n} - 1)(p^{2n} - 1)$. (D 31, 388.)

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These lists with the various alternating groups and the cyclic groups of prime order, give all known finite simple groups, with the exception of four due to M 70.

## ADDENDA

(For several of the titles below as well as for other very valuable assistance I am indebted to Prof. Miller.)

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| 17 | Gs, *for* MA, 17 *read* MA, 18. |
| 18 | Hd, *for* 84–76 *read* 84–86. |
| 20 | Hn, *for* Huntingdon *read* Huntington. |
| 21 | Kp, *for* 1885 *read* 1886. |
| 35 | Sd, *for* Séderberg *read* Söderberg. |
| ¶23 | Line 7 from end, *for* F1, 180, *read* W1, 302. |
| ¶29 | Line 2, *for* G4 *read* Ch 4. |
| ¶70 | Line 5, *for* Ft, 180 *read* F1, 180. |